# Jacobian Of Implicit Function

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(For Two functions, $f_1$ and $f_2$)

If $x, y, u, v$ are connected by Implicit functions

$f_1(x, y, u, v) = 0$ and $f_2(x, y, u, v) = 0$

Then $$\frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\left[\dfrac{\partial(f_1, f_2)}{\partial(x,y)}\right]}{\left[\dfrac{\partial(f_1, f_2)}{\partial(u,v)}\right]}$$

In General for 3 functions, $f_1$ $f_2$ and $f_3$

If $x, y, z, u, v, w$ are connected by Implicit functions

$f_1(x, y, z, u, v, w) = 0$, $f_2(x, y, z, u, v, w) = 0$

and $f_2(x, y, z, u, v, w) = 0$

Then $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \left[\frac{\dfrac{\partial(f_1, f_2, f_3)}{\partial(x,y,z)}}{\dfrac{\partial(f_1, f_2, f_3)}{\partial(u,v,w)}}\right]$$

Q:-1. If $x^2 + y^2 + u^2 + v^2 = 0$ and

$uv + xy = 0$, Then prove that

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{x^2 - y^2}{u^2 + v^2}$$

Sol:- let $f_1 = x^2 + y^2 + u^2 + v^2$

and $f_2 = uv + xy$

where $f_1$ and $f_2$ are Implicit functions of $x, y, u,$ and $v$,

$$\frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\dfrac{\partial(f_1, f_2)}{\partial(x,y)}}{} \quad \text{———— (1)}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\frac{\partial(x,y)}{\partial(f_1 f_2)}}{\partial(u,v)} \quad \text{———————} (1)$$

Now $\frac{\partial(f_1, f_2)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ y & x \end{vmatrix} = 2x^2 - 2y^2$

$= 2(x^2 - y^2)$

and $\frac{\partial(f_1 f_2)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ v & u \end{vmatrix} = 2u^2 + 2v^2$

$= 2(u^2 + v^2)$

using in equation (I), we get

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{2(x^2-y^2)}{2(u^2+v^2)} = \frac{x^2-y^2}{u^2+v^2}$$

Proved.

Q:-2 If $u^3 = xyz$, $\frac{1}{v} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$, $w^2 = x^2 + y^2 + z^2$

Then prove that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{-v(y-z)(z-x)(x-y)(x+y+z)}{3u^2 w (yz + zx + xy)}$$

Sol:- let $F_1 \equiv u^3 - xyz = 0$

$F_2 \equiv \frac{1}{v} - \frac{1}{x} - \frac{1}{y} - \frac{1}{z} = 0$

and $F_3 \equiv w^2 - x^2 - y^2 - z^2 = 0$

we know that $\frac{\partial(F_1, F_2, F_3)}{}$

we Know that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\dfrac{\partial(F_1,F_2,F_3)}{\partial(x,y,z)}}{\dfrac{\partial(F_1,F_2,F_3)}{\partial(u,v,w)}} \quad \text{—— (1)}$$

Now, $$\frac{\partial(F_1,F_2,F_3)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} \end{vmatrix} = \begin{vmatrix} -yz & -zx & -xy \\ \frac{1}{x^2} & \frac{1}{y^2} & \frac{1}{z^2} \\ -2x & -2y & -2z \end{vmatrix}$$

Taking $\frac{1}{x^2}$ from $C_1$, $\frac{1}{y^2}$ from $C_2$, $\frac{1}{z^2}$ from $C_3$ and $(-2)$ from $R_3$

$$= \frac{2}{x^2y^2z^2} \begin{vmatrix} x^2yz & y^2zx & z^2xy \\ 1 & 1 & 1 \\ x^3 & y^3 & z^3 \end{vmatrix}$$

$$= \frac{2xyz}{x^2y^2z^2} \begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ x^3 & y^3 & z^3 \end{vmatrix} \qquad \left\{ \text{Taking } xyz \text{ common from } R_1 \right\}$$

$$= \frac{2}{xyz} \begin{vmatrix} x & y-x & z-x \\ 1 & 0 & 0 \\ x^3 & y^3-x^3 & z^3-x^3 \end{vmatrix} \qquad \left[ \begin{array}{l} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{array} \right.$$

$$= \frac{2(y-x)(z-x)}{xyz}\begin{vmatrix} x & 1 & 1 \\ 1 & 0 & 0 \\ x^3 & y^2+x^2+yx & z^2+x^2+xz \end{vmatrix}$$

Expanding by $R_2$

$$= \frac{2(y-x)(z-x)}{xyz}\left\{-1\left[\left(z^2+\cancel{x^2}+xz-y^2-\cancel{x^2}-yx\right)\right]\right\}$$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

$$= -2\frac{(y-x)(z-x)}{xyz}\left[(z^2-y^2)+(xz-yx)\right]$$

$$= -2\frac{(y-x)(z-x)}{xyz}\left[(z-y)(z+y)+x(z-y)\right]$$

$$= -2\frac{(y-x)(z-x)}{xyz}\cdot(z-y)(x+y+z)$$

$$= \frac{-2(x-y)(y-z)(z-x)(x+y+z)}{xyz}$$

Now $\frac{\partial(F_1\ F_2\ F_3)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} & \frac{\partial F_1}{\partial w} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} & \frac{\partial F_2}{\partial w} \\ \frac{\partial F_3}{\partial u} & \frac{\partial F_3}{\partial v} & \frac{\partial F_3}{\partial w} \end{vmatrix} = \begin{vmatrix} 3u^2 & 0 & 0 \\ 0 & \frac{-1}{v^2} & 0 \\ 0 & 0 & 2w \end{vmatrix}$

$$= (3u^2)\left(-\frac{1}{v^2}\right)(2w)$$

$$= -6\frac{u^2w}{v^2}$$

Using in equation (1), we get

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)\left[\frac{\frac{-2(x-y)(y-z)(z-x)(x+y+z)}{xyz}}{-\frac{6u^2w}{v^2}}\right]$$

$$= \frac{-v(x-y)(y-z)(z-x)(x+y+z)}{3u^2w} \cdot \frac{v}{xyz}$$

$$\frac{1}{v} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

$$= \frac{xy+yz+zx}{xyz}$$

$$\Rightarrow xyz = \frac{xy+yz+zx}{v}$$

$$\Rightarrow \frac{v}{xyz} = \frac{1}{xy+yz+zx}$$

$$= \frac{-v(y-z)(z-x)(x-y)(x+y+z)}{3u^2w\,(yz+zx+xy)}$$

Proved.

H.W

Q:- If $u^3+v+w = x+y^2+z^2$

$u+v^3+w = x^2+y+z^2$

$u+v+w^3 = x^2+y^2+z$, Prove that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{1-4(xy+yz+zx)+16xyz}{2-3(u^2+v^2+w^2)+27u^2v^2w^2}.$$